Inverse Functions and Trigonometric Equations - Solution Key

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CHAPTER 1 Inverse Functions and Trigonometric Equations - Solution Key

CHAPTER OUTLINE

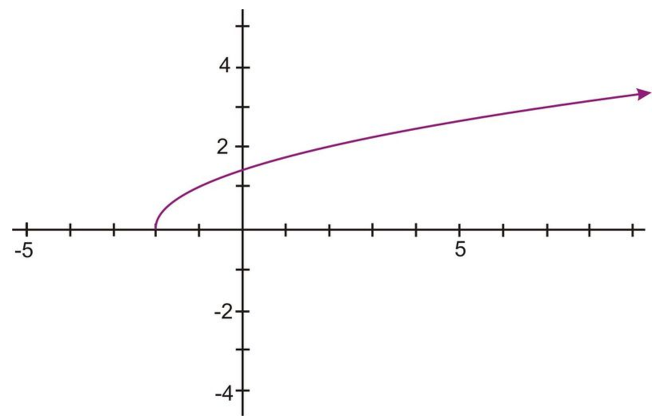
1.1 Inverse Functions and Trigonometric Equations

General Definitions of Inverse Trigonometric Functions

Review Exercises

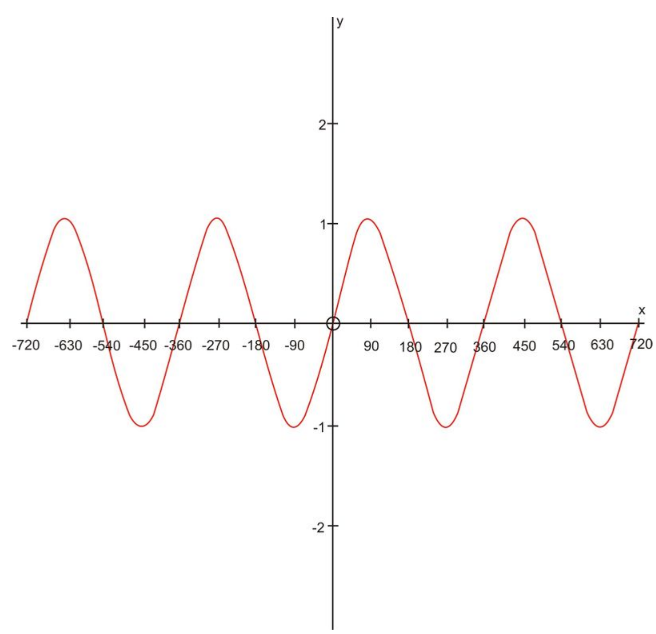
1.

a)



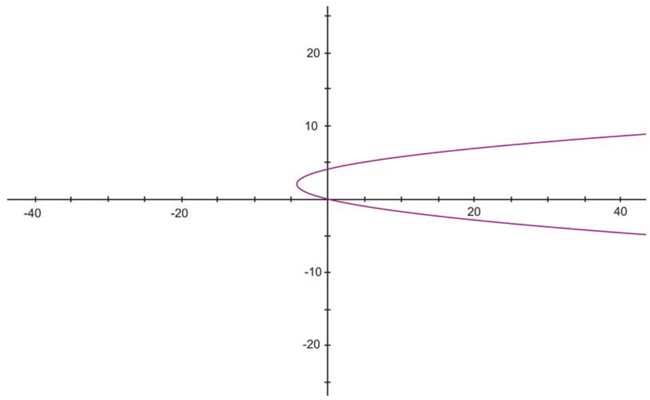
This graph represents a one-to-one function because a vertical line would cross the graph at only one point and a horizontal line would also cross the graph at only one point. Therefore the graph passes both the vertical line test and the horizontal line test. At this point students do know whether or not the function has an inverse that is a function. As a result, it is fine to accept whatever answer the students present as long as they justify their answer.

b)



This graph represents a function because it passes the vertical line test. However, the graph does not pass the horizontal line test. It does not have an inverse that is a function.

c)



The above graph passes the horizontal line test only. It fails the vertical line test. Therefore, this graph does not represent a one-to-one function. It does however, have an inverse that is a function.

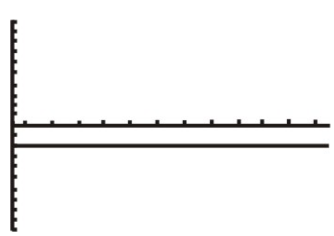
2. To calculate the measure of the angle that the ladder makes with the floor, the trigonometric ratio for cosine must be used. The ladder is the hypotenuse of the right triangle and the distance from the wall is the adjacent side with respect to the reference angle.

$$\begin{aligned}\cos\ \theta &= \frac{\text{adj}}{\text{hyp}}\\ \cos\ \theta &= \frac{4}{9}\\ \cos\ \theta &= 0.4444\\ \cos^{-1}(\cos^{-1}\ \theta) &= \cos^{-1} = (0.4444)\\ \theta &\approx 63.6^\circ\end{aligned}$$

1. $\sin^{-1}\left(\frac{\pi}{2}\right)$ does not exist. If π is considered as having an approximate value of 3.14, then $\frac{3.14}{2} \approx 1.57$. The domain of the sine function is $[-1,1]$.

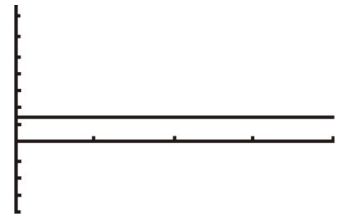
2. $\tan^{-1}(-1)$ does exist. The graph of $\tan^{-1}(-1)$ can be done on the graphing calculator. The exact value is $-\frac{\pi}{4}$.

$y = \tan^{-1}$



3. $\cos^{-1}\left(\frac{1}{2}\right)$ does exist. The graph of $\cos^{-1}\left(\frac{1}{2}\right)$ can be done on the graphing calculator. The exact value is $\frac{\pi}{3}$.

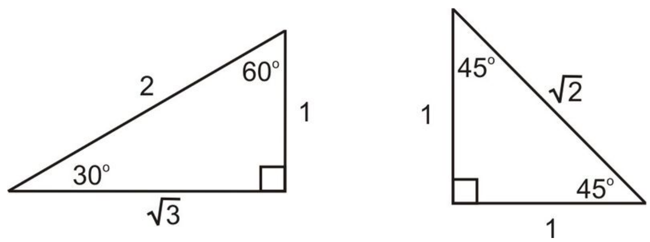
$y = \cos^{-1}\left(\frac{1}{2}\right)$



Ranges of Inverse Circular Functions

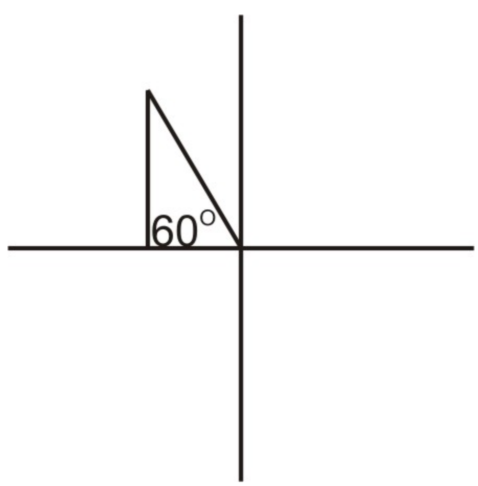
Review Exercises

To determine the exact values of the following functions, the special triangles may be used or the unit circle. The special triangles may be easier for students to sketch and the answers can be readily converted to radians or degrees if necessary.

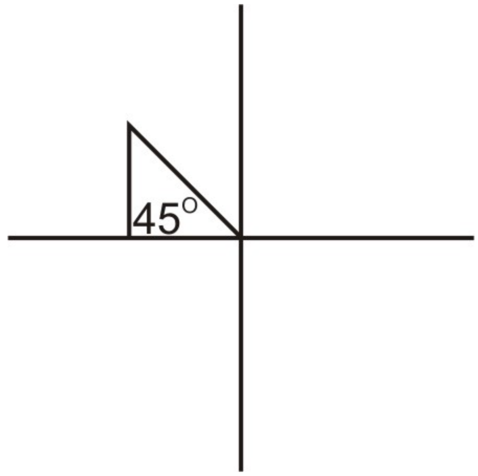


1.

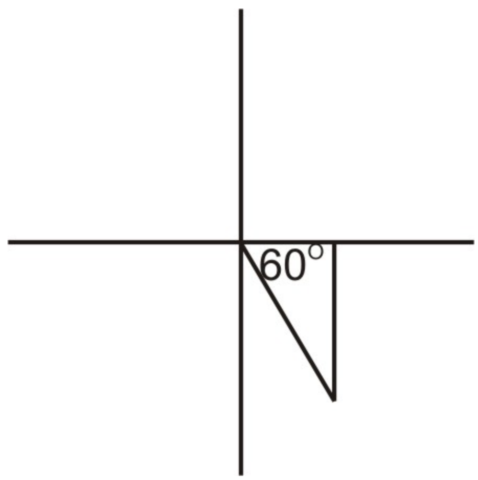
a) $\cos\ 120^\circ$ An angle of 120° has a related angle of 60° in the 2^{nd} quadrant. The cosine function is negative in this quadrant. Using the special triangle, the exact value of $\cos\ 120^\circ$ is $\frac{\text{adj}}{\text{hyp}} = -\frac{1}{2}$



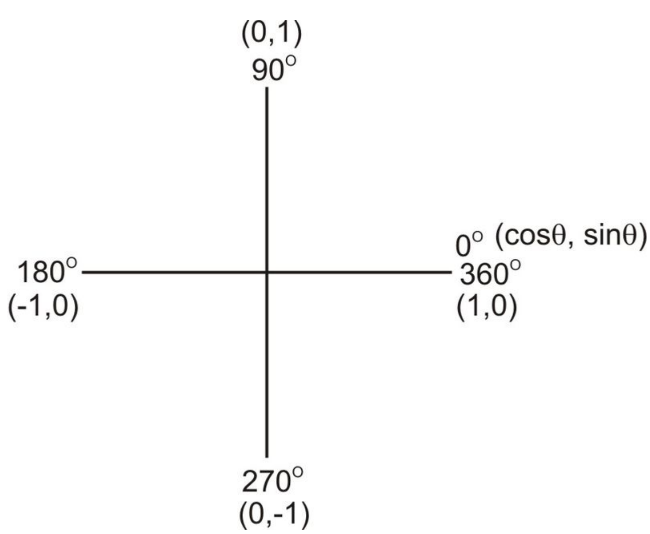
b) $\csc \frac{3\pi}{4}$. An angle of $\frac{3\pi}{4}$rad (135°) has a related angle of $\frac{\pi}{4}$rad (45°) in the 2^{nd} quadrant. Cosecant is the reciprocal of the sine function and is positive in the 2^{nd} quadrant. Therefore, using the special triangle, if $\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$ *then* $\csc \frac{3\pi}{4} = \sqrt{2}$.



c) $\tan \frac{5\pi}{3}$. An angle of $\frac{5\pi}{3}$ rad (300°) has a related angle of $\frac{\pi}{3}$ rad (60°) in the 4^{th} quadrant. The tangent function has a negative value in the 4^{th} quadrant. Using the special triangle, the exact value of $\tan \frac{5\pi}{3}$ is $\frac{\text{opp}}{\text{adj}} = -\frac{\sqrt{3}}{1}$.



a)



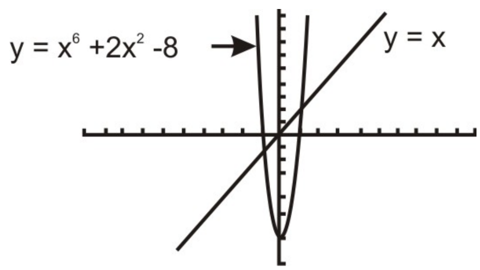
Using this diagram shows that $\cos^{-1}(0) = 90^\circ$ or $\frac{90^\circ}{180^\circ} = \frac{\pi}{2}$rad

b) $\tan^1(-\sqrt{3}) = -60^\circ$ in either the 2$^{\text{nd}}$ quadrant or the 4$^{\text{th}}$ quadrant since the tangent function is negative in these quadrants. The exact value of $\tan^1(-\sqrt{3})$ is $\tan^1(-\sqrt{3}) = -60^\circ$ or $-\frac{\pi}{3}$rad

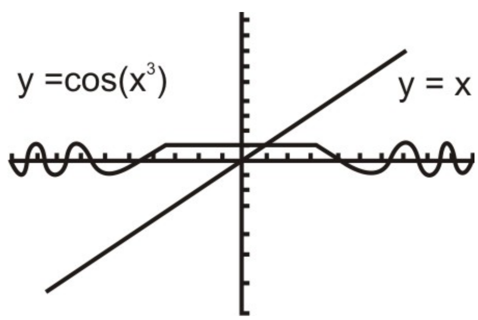
c) $\sin^{-1}\left(-\frac{1}{2}\right) = -30^\circ$ in either the 3$^{\text{rd}}$ or the 4$^{\text{th}}$ quadrant since the sine function is negative in these quadrants. The exact value of $\sin^{-1}\left(-\frac{1}{2}\right) = -30^\circ$ or $-\frac{\pi}{6}$rad is

Review Exercises

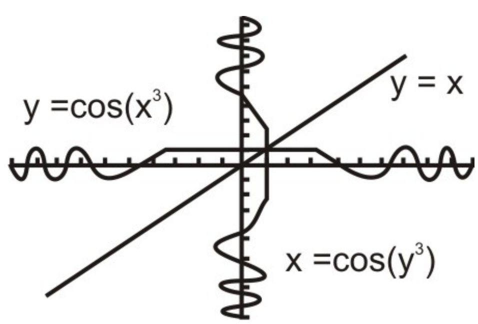
1. The graphs of $y = x^6 + 2x^2 - 8$ and $y = x$ can be graphed using the TI-83. From the graph, it is obvious that the graph of $y = x^6 + 2x^2 - 8$ would not reflect across the line $y = x$ as a mirror image. Therefore the function is not invertible.



b) The graphs of $y = \cos(x^3)$ *and* $y = x$ are shown below as displayed on the TI-83.



The graph of the inverse $x = \cos(y^3)$ is shown below as it appears when added to the above graph on the TI-83.



The function $y = \cos(x^3)$ is invertible because its inverse, $x = \cos(y^3)$, is the mirror image of $y = \cos(x^3)$ reflected across the line $y = x$.

2. To prove that the functions $f(x) = 1 - \frac{1}{x-1}$ and $f^{-1}(x) = 1 + \frac{1}{1-x}$ are inverses, prove algebraically that $f\left(f^{-1}(x)\right) = x$. and $f^{-1}(f(x)) = x$.

$$f(f^{-1}(x)) = 1 - \frac{1}{\left(1+\frac{1}{1-x}\right)-1} \rightarrow \text{common denominator}$$
$$f(f^{-1}(x)) = 1 - \frac{1}{\left(1\left(\frac{1-x}{1-x}\right)+\frac{1}{1-x}\right)-1} \rightarrow \text{simplify}$$
$$f(f^{-1}(x)) = 1 - \frac{1}{\frac{1-x+1}{1-x}-1} \rightarrow \text{simplify}$$
$$f(f^{-1}(x)) = 1 - \frac{1}{\frac{2-x}{1-x}-\left(\frac{1-x}{1-x}\right)1} \rightarrow \text{simplify} \rightarrow \text{common denominator}$$
$$f(f^{-1}(x)) = 1 - \frac{1}{\frac{2-x-1+x}{1-x}} \rightarrow \text{simplify}$$
$$f(f^{-1}(x)) = 1 - \frac{1}{\frac{1}{1-x}} \rightarrow \text{simplify}$$
$$f(f^{-1}(x)) = 1 - \left[1\left(\frac{1-x}{1}\right)\right] \rightarrow \text{simplify}$$
$$f(f^{-1}(x)) = 1-(1-x) \rightarrow \text{simplify}$$
$$f(f^{-1}(x)) = 1-1+x \rightarrow \text{simplify}$$
$$f(f^{-1}(x)) = x$$

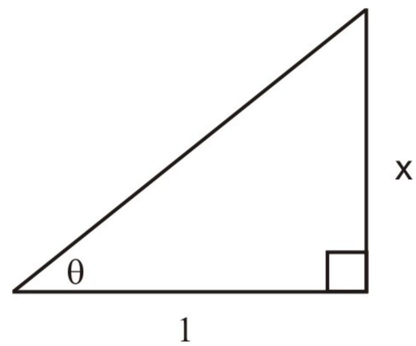
$$f^{-1}(f(x)) = 1 + \frac{1}{1-\left(1-\frac{1}{x-1}\right)} \rightarrow \text{common denominator}$$
$$f^{-1}(f(x)) = 1 + \frac{1}{1-\left(1\left(\frac{x-1}{x-1}\right)-\frac{1}{x-1}\right)} \rightarrow \text{simplify}$$
$$f^{-1}(f(x)) = 1 + \frac{1}{1-\left(\frac{x-1-1}{x-1}\right)} \rightarrow \text{simplify}$$
$$f^{-1}(f(x)) = 1 + \frac{1}{1\left(\frac{x-1}{x-1}\right)-\left(\frac{x-2}{x-1}\right)} \rightarrow \text{simplify} \rightarrow \text{common denominator}$$
$$f^{-1}(f(x)) = 1 + \frac{1}{\left(\frac{1}{x-1}\right)} \rightarrow \text{simplify}$$
$$f^{-1}(f(x)) = 1 + \left[1\left(\frac{x-1}{1}\right)\right] \rightarrow \text{simplify}$$
$$f^{-1}(f(x)) = 1+x-1 \rightarrow \text{simplify}$$
$$f^{-1}(f(x)) = x$$

Derive Properties of Other Five Inverse Circular Functions in Terms of Arctan

Review Exercises

1.

a)



Using this triangle will determine a value for $\tan^{-1}(x)$.

$$\begin{aligned}
\tan\ \theta &= \frac{\text{opp}}{\text{adj}} \\
\tan\ \theta &= \frac{x}{1} \\
\tan^{-1}\ (\tan\ \theta) &= \tan^{-1}(x) \\
\theta &= \tan^{-1}(x)
\end{aligned}$$

$\cos^2(\tan^{-1}x) = \cos^2(\theta)$ Using the same triangle, determine the length of the hypotenuse.

$$\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(h)^2 &= (x)^2 + (1)^2 \\
(h)^2 &= x^2 + 1 \\
\sqrt{(h)^2} &= \sqrt{x^2+1} \\
h &= \sqrt{x^2+1}
\end{aligned}$$

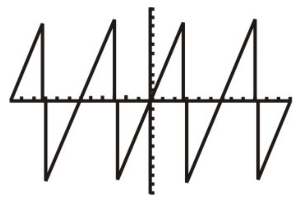
$$\begin{aligned}
\cos\ \theta &= \frac{\text{adj}}{\text{hyp}} \\
\cos\ \theta &= \frac{1}{\sqrt{x^2+1}} \\
\cos^2\ \theta &= \left(\frac{1}{\sqrt{x^2+1}}\right)^2 \\
\cos^2\ \theta &= \frac{1}{x^2+1} \\
\therefore\ \cos^2(\tan^{-1}x) &= \frac{1}{x^2+1}
\end{aligned}$$

b) $\cot(\tan^{-1}x^2) - \cot^2(\tan^{-1}x)$

As shown above, $\tan^{-1}x = \theta$

$$\cot\ \theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{x}$$
$$\cot^2\ \theta = \left(\frac{1}{x}\right)^2 = \frac{1}{x^2}$$
$$\therefore\ \cot(\tan^{-1} x^2) = \frac{1}{x^2}$$

2. The graph of can be displayed using the TI-83.



The domain is the set of all real numbers except $\frac{\pi}{2} + k\pi$ *where* k is an integer and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Review Exercises

1. To prove sin $\left(\left(\frac{\pi}{2}\right) - \theta\right) = \cos\ \theta$ the cofunction identities for sine and cosine must be used.

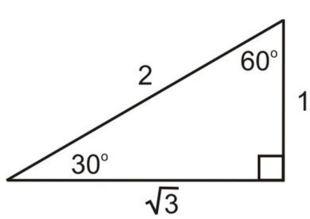
$$\sin\ \left(\frac{\pi}{2} - \theta\right) = \cos\ \theta \to \text{cofunction identities}$$
$$\cos\ \left(\frac{\pi}{2} - \theta\right) = \sin\ \theta$$
$$\sin\ \left(\left(\frac{\pi}{2}\right) - \theta\right) = \cos\ \left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right) \to \text{simplify}$$
$$\sin\ \left(\left(\frac{\pi}{2}\right) - \theta\right) = \cos\ \left(\frac{\pi}{2} - \frac{\pi}{2} + \theta\right) \to \text{simplify}$$
$$\sin\ \left(\left(\frac{\pi}{2}\right) - \theta\right) = \cos(0 + \theta)$$
$$\sin\ \left(\left(\frac{\pi}{2}\right) - \theta\right) = \cos(\theta)$$

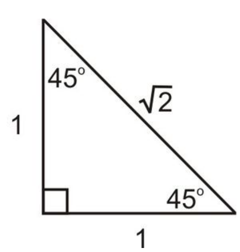
2. If sin $\left(\frac{\pi}{2} - \theta\right) = 0.68$ and sin $\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$ *then*

$$-\sin\ \left(\frac{\pi}{2} - \theta\right) = \cos(-\theta)$$
$$\therefore\ \cos(-\theta) = -0.68$$

Review Exercises

1. To determine the exact values of the following inverse functions, the special triangles can be used.



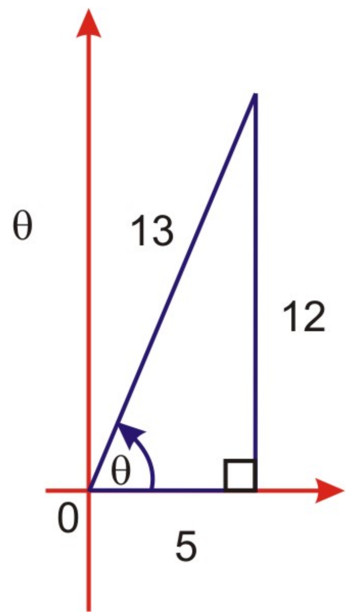


a) $\cos^{-1}\left(\sqrt{\frac{3}{2}}\right)$ From the triangles, it can be verified that $\cos\,\theta(30^\circ) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2}$. The exact value of $\cos^{-1}\left(\sqrt{\frac{3}{2}}\right)$ is $\frac{\pi}{6}$.

b) $\sec^{-1}(\sqrt{2})$. The secant function is the reciprocal of the cosine function. Therefore, $\sec\,\theta(45^\circ) = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{2}}{1}$. The exact value of $\sec^{-1}(\sqrt{2})$ is $\frac{\pi}{4}$.

c) $\sec^{-1}(-\sqrt{2})$. The secant function is the reciprocal of the cosine function and is therefore negative in the 2^{nd} and 3^{rd} quadrants. An angle of 45° in standard position in the 2nd quadrant is an angle of 225°. $\sec\theta(225^\circ) = \frac{\text{hyp}}{\text{adj}} = -\frac{\sqrt{2}}{1}$ The exact value of $\sec^{-1}(\sqrt{2})$ is $\frac{5\pi}{4}$.

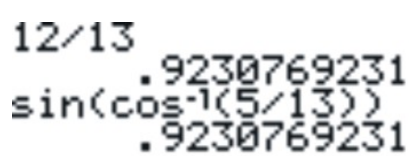
Review Exercises



1. To evaluate $\sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$, the angle is located in the 1^{st} quadrant. Working backwards, the previous line to $\cos^{-1}\left(\frac{5}{13}\right)$ is $\cos^{-1}(\cos\,\theta) = \cos^{-1}\left(\frac{5}{13}\right)$. Thus, $\cos\,\theta = \frac{5}{13}$.

$$\sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right) = \sin\,\theta$$
$$\sin\,\theta = \frac{12}{13}.$$

This solution can be verified using technology:



Revisiting

Revisiting $y = c + a\,\cos\,b(x - d)$

Review Exercises

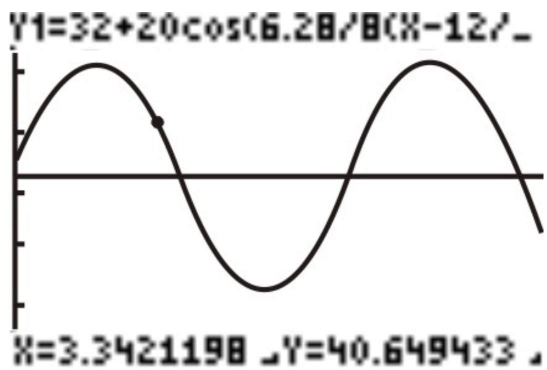
1. The transformations of $y = \cos\ x$ are the vertical reflection; vertical stretch; vertical translation; horizontal stretch and horizontal translation. These changes can be used to write the equation to model a graph of a sinusoidal curve. The simplest way to present these transformations is show them in a list.

$$V.R. = \text{No} \qquad V.S. = \frac{5 - \ -1}{2} = 3 \qquad V.T. = 2 \qquad H.S. = \frac{210^\circ - 30^\circ}{360^\circ} = \frac{1}{2} \qquad H.T. = 30^\circ$$

The equation that would model the graph of $y = \cos\ x$ that has undergone these transformations is $y = 3\ \cos(2(x - 30^\circ)) + 2$

Review Exercises

1. This problem is an example of an application of solving the equation $y = c + a\cos b(x - d)$ in terms of x. The problem that is presented should be sketched as a graph to facilitate obtaining an equation to model the curve. Once this has been done, the equation can then be entered into the TI-83 and the trace function can be used to estimate a value for x. The following graph was done on the calculator and it shows an estimate of 3.34 seconds for x.



$$y = 32 + \cos\frac{6.28}{8}\left(x - \frac{12}{6.28}\right) \rightarrow \text{equation}$$

$$y = 32 + \cos\frac{6.28}{8}\left(x - \frac{12}{6.28}\right) \rightarrow \text{solve for } x$$

$$x = \frac{\cos^{-1}\left[\frac{y-c}{a}\right]}{b} + d \rightarrow y = 40, c = 32, b = \frac{6.28}{8}, d = \frac{12}{6.28}$$

$$x = \frac{\cos^{-1}\left[\frac{40-32}{20}\right]}{\frac{6.28}{8}} + \frac{12}{6.28}\text{simplify}$$

$$x = \frac{\cos^{-1}\left[\frac{8}{20}\right]}{\frac{6.28}{8}} + \frac{12}{6.28} \rightarrow \text{simplify } u\ \sin\ g\ TI - 83$$

$$x \approx 3.39 \text{ seconds}$$

Solving Trigonometric Equations Analytically

Review Exercises

1. To solve the equation $\sin\ 2\theta = 0.6\ for\ 0 \le \theta < 2\pi$, involves determining all the possible values for $\sin\ 2\theta = 0.6\ for\ 0 \le \theta < 4\pi$ and then dividing these values by 2 to obtain the values for π. The angle is measured in radians since the domain is given in these units.

$$\begin{aligned}\sin\ 2\theta &= 0.6 \rightarrow \text{determine reference angle.}\\ \alpha &= \sin^{-1}(0.6)\\ \alpha &= 0.6435\end{aligned}$$

The angles for 2θ will be in quadrants $1, 2, 5, 6$.

$$\begin{aligned}2\theta &= 0.6435, \pi - 0.6435, 2\pi + 0.6435, 3\pi - 0.6435\\ 2\theta &= 0.6435, 2.4980, 6.9266, 8.7812\end{aligned}$$

The angles for θ in the domain $[0, 2\pi)$ are:

$$\theta = 0.3218, 1.2490, 3.4633, 4.3906$$

It is not necessary, but these results can be confirmed by using the TI-83 calculator to graph the function.

2. To solve the equation $\cos^2\ x = \frac{1}{16}$ over the interval $[0, 2\pi)$ involves applying the fact that the square root of a number can be positive or negative. This will allow the equation to be solved for all possible values.

$$\begin{aligned}\cos^2\ x &= \frac{1}{16} \rightarrow \sqrt{\text{Both sides}}\\ \sqrt{\cos^2\ x} &= \sqrt{\frac{1}{16}} \rightarrow \text{simplify}\\ \cos\ x &= \pm\frac{1}{4}\\ \cos^{-1}(\cos x) &= \cos^{-1}\left(\frac{1}{4}\right)\end{aligned}$$

Then

$$\begin{aligned}x &= 1.3181\text{radians} \rightarrow \text{1st eqadrant}\\ x &= 2\pi - 1.3181\\ x &= 4.9651\text{radians} \rightarrow \text{4th eqadrant}\\ \cos^{-1}(\cos x) &= \cos^{-1}\left(-\frac{1}{4}\right)\end{aligned}$$

Or

$$\begin{aligned}
x &= 1.8235\text{radians} \rightarrow \text{1st eqadrant} \\
x &= 2\pi - 1.8235 \rightarrow \text{3rd eqadrant} \\
x &= 4.4597\text{radians}
\end{aligned}$$

Once again, the results can be confirmed by graphing the function using the TI-83.

3. To solve the equation $\sin\ 4\theta - \cos\ 2\theta = 0$ for all values of θ such that $0 \le \theta \le 2\pi$ involves using the double angle identity for sine.

$$\begin{aligned}
\sin\ 4\theta - \cos\ 2\theta &= 0 \\
2\sin\ 2\theta\ \cos\ 2\theta - \cos\ 2\theta &= 0 \rightarrow \text{common factor} \\
\cos\ 2\theta(2\ \sin\ 2\theta - 1) &= 0 \rightarrow \text{simplify}
\end{aligned}$$

Then $\cos\ 2\theta = 0$over the interval $[0, 4\pi]$

$$\begin{aligned}
2\theta &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \rightarrow \div 2 \\
\theta &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}
\end{aligned}$$

Or

$$\begin{aligned}
2\ \sin\ 2\theta - 1 &= 0 \\
2\ \sin\ 2\theta &= 1 \\
\sin\ 2\theta &= \frac{1}{2} \rightarrow \text{over the interval } [0, 4\pi] \\
2\theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \rightarrow \div 2 \\
\theta & \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}
\end{aligned}$$

Once again, the results can be confirmed by graphing the function using the TI-83.

4. To solve the equation $\tan\ 2x - \cot\ 2x = 0$ over the interval $0^\circ \le x < 360^\circ$ will involve using the reciprocal identity for cotangent and applying the fact that the square root of a number can be positive or negative. This will allow the equation to be solved for all possible values.

$$\begin{aligned}
\tan\ 2x - \cot\ 2x &= 0 \\
\tan\ 2x - \cot\ 2x &= 0 \rightarrow \cot\ x = \frac{1}{\tan\ x} \\
\tan\ 2x - \frac{1}{\tan\ 2x} &= 0 \rightarrow \text{simplify}
\end{aligned}$$

$$\begin{aligned}
\tan\ 2x(\tan\ 2x)-(\tan\ 2x)\frac{1}{\tan\ 2x} &= (\tan\ 2x)0 \rightarrow \text{simplify}\\
\tan\ 2x(\tan\ 2x)-\cancel{(\tan\ 2x)}\frac{1}{\cancel{\tan\ 2x}} &= (\tan\ 2x)0 \rightarrow \text{simplify}\\
\tan^2\ 2x-1 &= 0 \rightarrow \text{simplify}\\
\tan^2\ 2x &= 1 \rightarrow \sqrt{\text{Both sides}}\\
\sqrt{\tan^2\ 2x} &= \sqrt{1}\\
\tan\ 2x &= \pm 1
\end{aligned}$$

Then $\tan\ 2x = 1$ over the interval $[0^\circ, 720^\circ)$. The tangent function is positive in the $1^{\text{st}}, 3^{\text{rd}}, 5^{\text{th}}$ and 7th quadrants.

$$\begin{aligned}
2x &= 45^\circ, 225^\circ, 405^\circ, 5825^\circ \rightarrow \div 2\\
x &= 22.5^\circ, 112.5^\circ, 202.5^\circ, 292.5^\circ
\end{aligned}$$

Or $\tan\ 2x = -1$ over the interval $[0^\circ, 720^\circ)$. The tangent function is negative in the $2^{\text{nd}}, 4^{\text{th}}$, 6th, and 8th quadrants.

$$\begin{aligned}
2x &= 135^\circ, 315^\circ, 495^\circ, 675^\circ\\
x &= 67.5^\circ, 157.5^\circ, 247.5^\circ, 337.5^\circ
\end{aligned}$$

Once again, the results can be confirmed by graphing the function using the TI-83.

Review Exercises

1. To solve $\sin^2\ x - 2\sin\ x - 3 = 0$ for the values of x that are within the domain of the sine function, involves factoring the quadratic equation and determining the values that fall within the domain of $[0, 2\pi]$ *or* $[0, 360^\circ]$.

$$\begin{aligned}
\sin^2\ x - 2\sin\ x - 3 &= 0\\
\sin^2\ x - 2\sin\ x - 3 &= 0 \rightarrow \text{factor}\\
(\sin\ x+1)(\sin\ x-3) &= 0 \rightarrow \text{simplify}
\end{aligned}$$

Then

$$\begin{aligned}
\sin\ x+1 &= 0\\
\sin\ x &= -1\\
\sin^{-1}(\sin\ x) &= \sin^{-1}(-1)\\
x &= 270^\circ \text{ or } \frac{3\pi}{2}
\end{aligned}$$

Or

$$\begin{aligned}
\sin\ x-3 &= 0\\
\sin\ x &= 3
\end{aligned}$$

Does not exist. It is not in the range $[-1,1]$ of the sine function.

2. To solve the equation $\tan^2 x = 3\tan x$ for the principal values of x involves factoring the quadratic equation and determining the values that fall within the domain of the function.

$$\begin{aligned}
\tan^2 x &= 3 \tan x \\
\tan^2 x - 3 \tan x &= 0 \rightarrow \text{common factor} \\
\tan x(\tan x - 3) &= 0 \rightarrow \text{simplify}
\end{aligned}$$

Then

$\tan x = 0$

$\tan x(\tan x) = \tan^{-1}(0)$

$x = 0^\circ$

Or

$\tan x - 3 = 0$

$\tan x = 3$

$\tan^{-1}(\tan x) = \tan^{-1}(3)$

$x = 71.5^\circ$

3. To solve the equation $\sin x = \cos \frac{x}{2}$ over the interval $[0^\circ, 360^\circ)$ requires the use of the Pythagorean Identity $\sin^2 \theta + \cos^2 \theta = 1$ and the half-angle identity for cosine.

$$\begin{aligned}
\sin x &= \cos \frac{x}{2} \\
\sin x &= \cos \frac{x}{2} \rightarrow \pm\sqrt{\frac{\cos x + 1}{2}} \\
\sin x &= \pm\sqrt{\frac{\cos x + 1}{2}} \rightarrow \text{squre both sides} \\
(\sin x)^2 &= \left(\pm\sqrt{\frac{\cos x + 1}{2}}\right)^2 \rightarrow \text{squre both sides} \\
\sin^2 x &= \frac{\cos x + 1}{2} \rightarrow \sin^2 x + \cos^2 x = 1 \\
1 - \cos^2 x &= \frac{\cos x + 1}{2} \rightarrow \text{simplify} \\
2(1 - \cos^2 x) &= 2\left(\frac{\cos x + 1}{2}\right) \rightarrow \text{simplify} \\
2(1 - \cos^2 x) &= \cancel{2}\left(\frac{\cos x + 1}{\cancel{2}}\right) \rightarrow \text{simplify} \\
2 - 2\cos^2 x &= \cos x + 1 \rightarrow \text{simplify}
\end{aligned}$$

$$2 - 2\cos^2 x - \cos x - 1 = 0 \rightarrow \text{simplify}$$

$$\begin{aligned}
-2\cos^2 x - \cos x + 1 &= 0 \rightarrow \div(-1) \\
2\cos^2 x + \cos x - 1 &= 0 \rightarrow \text{factor} \\
(2 \cos x - 1)(\cos x + 1) &= 0
\end{aligned}$$

Then

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$\cos^{-1}(\cos x) = \cos^{-1}\left(\frac{1}{2}\right)$

Cosine is positive in the 1st and 4th quadrants.

$x = 60^\circ, 300^\circ$

Or

$\cos x + 1 = 0$

$\cos x = -1$

$\cos^{-1}(\cos x) = \cos^{-1}(-1)$

Cosine is negative in the 2nd and 3rd quadrant.

$x = 180^\circ$

4. To solve the equation $3 - 3\sin^2 x = 8\sin x$ over the interval $[0, 2\pi]$ requires factoring the quadratic equation and solving for all the solutions.

$$
\begin{aligned}
3 - 3\sin^2 x &= 8\sin x \\
3 - 3\sin^2 x - 8\sin x &= 8\sin x - 8\sin x \rightarrow \text{simplify} \\
-3\sin^2 x - 8\sin x + 3 &= 0 \rightarrow \div(-1) \\
3\sin^2 x + 8\sin x - 3 &= 0 \rightarrow \text{factor} \\
(3\sin x - 1)(\sin x + 3) &= 0
\end{aligned}
$$

Then

$3\sin x - 1 = 0$

$\sin x = \frac{1}{3}$

$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{1}{3}\right)$

Sine is positive in the 1st and 2nd quadrants.

$x = 0.3398\text{radians}$

$x = \pi - 0.3398$

$x = 2.8018\text{radians}$

Or

$\sin x + 3 = 0$

$\sin x = -3$

$\sin^{-1}(\sin x)\sin^{-1}(-3)$

Does not exist.

Review Exercises

1. To solve the equation $2\sin x \tan x = \tan x + \sec x$ for all values of $x \varepsilon [0, 2\pi]$ requires the use of the quotient identity for tangent and the reciprocal identity for secant.